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ALGORITHM OF CALCULATION OF THE
TWO-DIMENSIONAL DISCRETE FOURIER
TRANSFORM WITH EQUAL ORDERS

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A general, for any orders equal power of prime numbers, and
optimal, by the number of used one-dimensional discrete Fourier
transforms (DFT), algorithm of the two-dimensional discrete
Fourier transform (TDFT) is proposed, by means of the tensor
orthogonal transformation.

Consider arbitrary two-dimensional discrete signal \( \{f_{n,m}\} \) of size \( N \times N \), where \( N = L^r \) is a power of a prime number. Due to the tensor
representation [1], each component of the Fourier transform \( F_{p,s} \) of the
signal at sample \( (p, s) \) is described uniquely by the \( N \)-dimensional vector
\( \mathbf{F}_{p,s} = (f_{p,s,1}, \ldots, f_{p,s,N}) \), for which the following holds:

\[
F_{kp,ks} = \sum_{n,m=1}^{N} f_{n,m} W^{nkp+ms} = \sum_{t=1}^{N} f_{p,s,t} W^{kt}, \quad k = 1 \div N, \quad (1)
\]

where \( W = W_N = \exp(2\pi i/N) \), and \( \overline{l} = l \mod N \) for arbitrary \( l \).

Since it is not possible to cover the domain of definition of the spectrum
\( X_N = [1, N] \times [1, N] \) by groups \( T_{p,s} = \{(kp, ks); k = 1 \div N\} \) in the
optimal way, i.e. when only the minimum intersections \{\( (N, N) \)\} take
place between the covering groups \( T_{p,s} \), then as in the case of \( L = 2 \) [1],
it is necessary to modify the tensor representation, in order to construct
an optimal covering of \( X_N \) by disjoint subsets of the groups \( T'_{p,s} \), which
will be the most suitable covering for calculating the \( N \times N \)-point two-
dimensional DFT.

It is not difficult to show, that the set \( J_N \) of samples \( (p, s) \), the corre-
sponding groups $T_{p,s}$ of which cover $X_N$ with a minimum total number of intersections between themselves can be taken equal

$$J_N = \bigcup_{s=1}^{N} (1, s) \cup \bigcup_{p=1}^{N/L} (Lp, 1)$$

(2)

with cardinality $\text{card} J_N = L^{r-1}(L + 1)$. Therefore, for calculating the $N \times N$-point two-dimensional DFT, it is enough to perform $\text{card} J_N$ separate $N$-point DFT, by means of formula (1).

If we define now, for arbitrary $(p, s) \in X_N$ and $t = 1 \div L^{r-1}$, the components

$$f'_{p,s,t} = \sum_{k=0}^{L-1} f_{p,s,t+kL^{r-1}} W^k_L, \quad W_L = \exp(2\pi i/L),$$

(3)

then, it follows from (1), that for all numbers of form $kL + 1$, where $k = 0 \div L^{r-1} - 1$, the following formula of calculation of the spectrum is valid:

$$F_{(kL+1)p,(kL+1)s} = \sum_{t=1}^{L^{r-1}} \left(f'_{p,s,t}W^t_L\right) W^{kt}_{L^{r-1}}.$$  

(4)

Therefore, the $L^{r-1}$-point DFT over the vector $\overline{G}_{p,s} = (g_{p,s,1}, \ldots, g_{p,s,L^{r-1}})$, that corresponds to $F_{p,s}$, where $g_{p,s,t} = f'_{p,s,t}W^t, t = 1 \div L^{r-1}$, defines the original spectrum of the signal at all points of the set

$$T'_{p,s} = \left\{(kL+1)p, (kL+1)s; k = 0 \div L^{r-1} - 1\right\}.$$  

(5)

The representation of the spectrum of the signal by the tensor of the 3rd order $\{f'_{p,s,t}\}$, which we call $L$-paired, allows for performing an effective calculation of the spectrum by (4), since the following optimal covering of the domain of definition of the spectrum holds:

$$X_N = \sum_{A_N}^{T'_{p,s}} \left\{(N, N)\right\}, \quad A_N = \sum_{n=0}^{r-1} \sum_{k=1}^{L-1} kL^n J_{L^{r-n}}.$$  

(6)
where the signs $\sum^*$ and $\dagger$ denote the union of disjoint sets. At that, if $(p, s) \in kL^nJ_{L^{r-n}}, \ n \in [0, r-1], \ k = 1 \div L-1$, then from the elementary properties of the $L$-paired representation it follows that in (4) we have

$$F_{(kL+1)p,(kL+1)s}^{L^{r-n-1}} = \sum_{l=1}^{L} g_{p,s,t} L^n W_{L^{r-n-1}}^{kt} . \quad (7)$$

Since $\text{card } A_N = (N-1)(L+1)$, it follows from (7) and (6) that for calculating the $N \times N$-point two-dimensional DFT it is enough to fulfill $(N-1)(L+1)$ separate DFTs, card $J_{L^{r-n}} \times (L-1)$ of which compose the $L^{r-n-1}$-point DFTs, for all $n = 0 \div r-1$.

Consequently, the lower estimation for the number $v_{N,N}$ of operations of multiplication by exponential factors necessary for calculation of the $N \times N$-point two-dimensional DFT can be defined by the formula

$$v_{N,N} = L^{r-1}(L^2-1) \sum_{n=0}^{r-1} L^{-n} v_{L^{r-n-1}} + N^2 - 1,$$

where $v_{L^{r-n-1}}$ denotes the minimum number of such operations for the $L^{r-n-1}$-point DFT, respectively, $n = 0 \div r-1$.

It should be noted, that the $L$-paired representation of the $L^r \times L^r$-point two-dimensional DFT is characterized by non-separable orthogonal tensor transformation $\chi'$ whose basic functions are

$$\chi'_{p,s,t} = \sum_{k=0}^{L-1} W_k^L \chi_{p,s,t+kL^{-1}},$$

where $\chi_{p,s,t}$ are the characteristic functions of the sets $V_{p,s,t} = \{(n,m); \ \bar{n}p + ms = t\}, \ \text{for all } p, s, t \in [1, N]$, which takes $L-1$ different nonzero complex values $W_1^L, \ldots, W_{L-1}^L$.

Therefore, all vectors $G_{p,s}, \ (p, s) \in A_N$, over which it is necessary to fulfill the corresponding DFTs, are obtained by the simple way from the result of this orthogonal transform over the original signal. Thus,
the tensor orthogonal transformation performs a splitting of the two-dimensional DFT on the minimum number of separable DFTs and allows, hence, for effective realization of the described algorithm of the two-dimensional spectrum of the signal.

REFERENCES


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