III. Direction Images

The representation of the image as the set of splitting-signals leads to the image decomposition by direction images. We first consider the tensor representation \[1\], \[6\]

\[
\{f_{n,m}\} \rightarrow \{f_{T_{p,s}}; (p, s) \in J_{N,N}\},
\]

where the splitting-signals \(f_{T_{p,s}} = \{f_{p,s,0}, f_{p,s,1}, f_{p,s,2}, \ldots, f_{p,s,N-1}\}\), are generated by frequencies from the given set \(J_{N,N}\) for the case of images \(N \times N\), when \(N\) is a prime. Then, we consider the cases when \(N\) is a power of two and odd prime. For these cases, the direction images will be defined by the paired representation \[2\]-\[9\].

A. Inverse Tensor transform

The tensor transform can be used for calculating and processing the 2-D DFT of the image, since each of splitting-signal \(f_{T_{p,s}}\) defines the 2-D DFT of the image at frequency-points of the cyclic group \(T_{p,s}\). The image can thus be calculated from the tensor transform by using the inverse 2-D DFT. The image can also be reconstructed directly from its tensor transform, without calculating the inverse 2-D DFT. The splitting-signals define unique direction components of the image.

Indeed, for a given generator \((p, s) \neq (0, 0)\), we consider the following incomplete 2-D DFT with values at frequency-points of the group \(T_{p,s}\): \[1\]

\[
D_{p_1,p_2} = D_{(p,s)}_{p_1,p_2} = \begin{cases} F_{p_1,p_2}; & \text{when } (p_1,p_2) = (\overline{kp},\overline{ks}), \text{ for } k \in \{0,1,\ldots,N-1\}, \\ 0, & \text{otherwise}, \end{cases}
\]

where \(p_1, p_2 = 0 : (N - 1)\). The inverse 2-D DFT of the incomplete transform \(D_{p_1,p_2}\) is the direction image \(d_{n_1,n_2}\), which can be calculated as follows:

\[
d_{n_1,n_2} = d_{(p,s)}_{n_1,n_2} = (\mathcal{F}^{-1}_{N,N} \circ D_{p,s})_{n_1,n_2} = \frac{1}{N^2} \sum_{p_1=0}^{N-1} \sum_{p_2=0}^{N-1} D_{p_1,p_2} W^{n_1 p_1 + n_2 p_2}
= \frac{1}{N^2} \sum_{(p_1,p_2) \in T_{p,s}} D_{p_1,p_2} W^{n_1 p_1 + n_2 p_2} = \frac{1}{N^2} \sum_{k=0}^{N-1} F_{kp,ks} W^{(kn_1 p_1 + n_2 p_2)}
= \frac{1}{N} \left[ \frac{1}{N} \sum_{k=0}^{N-1} F_{kp,ks} W^{k(n_1 p_1 + n_2 p_2)} \right] = \frac{1}{N} f_{p,s,(n_1 p_1 + n_2 p_2) \text{ mod } N}.\]
The interesting property of the tensor transform is derived. The direction image is composed by \( N \) values of the splitting-signal, which are placed at all points of the image \( N \times N \) along the parallel lines. As an example, Figure 1 shows the tree image of size 257 \( \times \) 257 in part (a), along with the splitting-signal generated by the frequency \((p, s) = (1, 5)\) in b. The 1-D DFT of the splitting-signal in absolute scale and the location of all frequency-points of the cyclic group \( T_{1,5} \), at which the 2-D DFT of the image is defined by this splitting-signal, are shown in c. The direction image \( d_{n_1, n_2}^{(1,5)} \) is illustrated in d.

The directional images of the tree image, which are generated by the frequencies \((p, s) = (0, 1), (1, 1), (1, 2), (1, 3), (1, 4), \) and \((1, 6)\) are shown in Figure 2 in parts a-f, respectively.

Fig. 1. (a) The image 257 \( \times \) 257, (b) the splitting-signal \( \{f_{1.5, t}; t = 0 : 256\} \), and (c) the 1-D DFT of the splitting-signal and frequency-points of \( T_{1,5} \), and (d) the corresponding direction image \( d_{n_1, n_2} \) (the image has been scaled).
Fig. 2. (a)-(f) Six direction images of the tree image in tensor representation, which are defined by frequencies $(p, s) = (0, 1), (1, 1), (1, 2), (1, 3), (1, 4)$, and (1, 6). (All images have been scaled.)

Cyclic groups $T_{p,s}$ intersect only in one point $(0, 0)$. The covering of the lattice $X_{N,N}$ is defined by $(N + 1)$ cyclic groups. Therefore, the sum of all $(N + 1)$ incomplete 2-D DFTs $D_{p_1,p_2}^{(p,s)}$, $(p, s) \in J_{N,N}$, equals the 2-D DFT of the image plus $N$ times single values of the $F_{0,0}$ at point $(0, 0)$. We denote by $O_{p_1 p_2}$ the complex matrix with all zero coefficients, except the first coefficient which equals $O_{0,0} = F_{0,0}$. The inverse 2-D DFT of the sum of all $(N + 1)$ incomplete 2-D DFTs can be calculated as follows:

$$
\sum_{(p,s) \in J_{N,N}} d_{n_1,n_2}^{(p,s)} = \sum_{(p,s) \in J_{N,N}} (F_{N,N}^{-1} \circ D_{p_1,p_2}^{p,s})_{n_1,n_2} = F_{N,N}^{-1} \circ \left( \sum_{(p,s) \in J_{N,N}} D_{p_1,p_2}^{p,s} \right)_{n_1,n_2}
$$

$$
= (F_{N,N}^{-1} \circ (F_{p_1,p_2} + N O_{p_1,p_2}))_{n_1,n_2}
$$

$$
= (F_{N,N}^{-1} \circ F_{p_1,p_2})_{n_1,n_2} + N (F_{N,N}^{-1} \circ O_{p_1,p_2})_{n_1,n_2}
$$

$$
= f_{n_1,n_2} + \frac{1}{N} F_{0,0} = f_{n_1,n_2} + NE[f].
$$

(3)

Here $E[f]$ denotes the mean of the image,

$$
E[f] = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} = \frac{1}{N^2} F_{0,0} = \frac{1}{N^2} f_{0,0,0}.
$$
We obtain the following formula of the inverse tensor transformation:

\[
f_{n_1,n_2} = \sum_{(p,s) \in J_{N,N}} d_{n_1,n_2}^{(p,s)} - NE[f] = \frac{1}{N} \sum_{(p,s) \in J_{N,N}} f_{p,s,(n_1p+n_2s) \mod N} - NE[f].
\] (4)

The set \(J_{N,N}\) is a set of \((N+1)\) generators \((p, s)\) which defines the irreducible covering of the lattice \(N \times N\) by groups \(T_{p,s}\). Considering the set \(J_{N,N}\) as

\[
J_{N,N} = \{(1, 0), (1, 1), (1, 2), (1, 3), ..., (1, N - 1)\} \cup \{(0, 1)\},
\]

we obtain the following statement.

**Statement 1: (Superposition by direction images)** The image \(f_{n_1,n_2}\) of size \(N \times N\), where \(N > 2\) is a prime, can be composed by \((N+1)\) direction images as follows:

\[
f_{n_1,n_2} = \left[ \sum_{s=0}^{N-1} d_{n_1,n_2}^{(1,s)} + d_{n_1,n_2}^{(0,1)} \right] - NE[f]
\]

\[
= \frac{1}{N} \left[ \sum_{s=0}^{N-1} f_{1,s,(n_1s+n_2) \mod N} + f_{0,1,n_2} \right] - NE[f],
\] (5)

\[
n_1, n_2 = 0 : (N - 1).
\]

If we define the set of generators as

\[
J_{N,N} = \{(0, 1), (1, 1), (2, 1), (3, 1), ..., (N - 1, 1)\} \cup \{(1, 0)\},
\]

the following composition is valid:

\[
f_{n_1,n_2} = \left[ \sum_{p=0}^{N-1} d_{n_1,n_2}^{(p,1)} + d_{n_1,n_2}^{(1,0)} \right] - NE[f]
\]

\[
= \frac{1}{N} \left[ \sum_{p=0}^{N-1} f_{p,1,(n_1p+n_2) \mod N} + f_{1,0,n_1} \right] - NE[f].
\] (6)

The image is the sum of components of \((N+1)\) splitting-signals, or the sum of \((N+1)\) direction images of the tensor representation of the image. The above inverse formulas for the image reconstruction from its tensor transform requires only two operations of multiplication and division by \(N\). The division by \(N\) can be considered in the definition of the tensor transform components. Then, the reconstruction requires only \(N^2\) such operations.
B. Inverse paired transform

Consider the image \( f_{n,m} \) of size \( N \times N \), when \( N = 2^r, r > 1 \). In paired representation, the image is the set of \( 3N - 2 \) splitting-signals

\[
\{ f'_{p,s,0}, f'_{p,s,2^n}, f'_{p,s,2^{2n}}, \ldots, f'_{p,s,N/2-2^n} \}
\]

which are generated by the frequencies \((p, s)\) from the set

\[
J'_{N,N} = \bigcup_{n=0}^{r-1} 2^n J_{N/2^n,N/2^n} \cup \{(0,0)\},
\]

where the subsets

\[
J_{N/2^n,N/2^n} = \{2^n(p,1); p = 0 : (N/2^n - 1)\} \cup \{2^n(1,2s); s = 0 : (N/2^{n+1} - 1)\}.
\]

The elements \( f'_{p,s,t} \) of the paired transform are calculated by

\[
f'_{p,s,t} = f_{p,s,t} - f_{p,s,t+N/2}, \quad t = 0 : (N/2 - 1).
\]

The paired transform is unitary and the paired splitting-signals define the corresponding direction image components, which together compose the image \( \{ f_{n,m} \} \). To derive such a composition, we take a generator \((p, s)\) and denote \( 2^k = \gcd(p, s) \). Let \( D \) be the 2-D DFT composed only from the components of the 2-D DFT with frequency-points of the subset \( T'_{p,s} \),

\[
D_{p_1,s_1} = \begin{cases} 
F_{p_1,s_1}; & \text{if } (p_1,s_1) \in T'_{p,s} \\
0; & \text{otherwise.}
\end{cases}
\]

The image of this incomplete 2-D transform is composed only by values of the splitting-signal generated by \((p, s)\). Let us assume that for this generator \( k \in \{1, 3, ..., r - 1\} \). The number of such generators is equal to \( 3N/2^{k+1} \). The calculation of the inverse transform of the defined
incomplete 2-D DFT results in the following direction image:

\[
d_{n,m} = d_{n,m;p,s} = \frac{1}{N^2} \sum_{p_1=0}^{N-1} \sum_{s_1=0}^{N-1} D_{p_1,s_1} W^{-(n p_1 + m s_1)}
\]

\[
= \frac{1}{N^2} \sum_{(p_1,s_1) \in T_{p,s}} F_{p_1,s_1} W^{-(n p_1 + m s_1)}
\]

\[
= \frac{1}{N^2} \sum_{l=0}^{N/2^{k+1}-1} F(2l+1)p(2l+1)s W^{-(2l+1)(np+ms)}
\]

\[
= \frac{1}{2^{k+1}N} \left( \frac{2^{k+1}}{N} \sum_{l=0}^{N/2^{k+1}-1} F(2l+1)p(2l+1)s W^{-lt} \right) W^{-t}
\]

\[
= \frac{1}{2^{k+1}N} f'_{p,s,(np+ms) \mod N}
\]

where we denoted \(2^k = (np + ms) \mod N\).

For the last generator is \((p,s) = (0,0)\), and \(T'_{0,0} = \{(0,0)\}\). The 2-D transform \(D\) is zero at all frequency-points except the point \((0,0)\), where the value of the transform equals \(F_{0,0}\).

In this case, we have the following constant-image:

\[
d_{n,m} = d_{n,m;0,0} = \frac{1}{N^2} F_{0,0} = \frac{1}{N^2} \sum_{n_1=0}^{N-1} \sum_{m_1=0}^{N-1} f_{n_1,m_1} = \frac{1}{N} E[f] = \frac{1}{N^2} f'_{0,0,0},
\]

where \(E[f]\) is the mean value of the image.

All \((3N-2)\) subsets \(T'_{p,s}\), with generators \((p,s)\) from the set \(J'_{N,N}\) compose a partition of the grid \(N \times N\). It means that the sum of corresponding \((3N-2)\) incomplete 2-D DFTs equals the 2-D DFT of the original image. In other words, the sum of all direction images \(d_{n,m;p,s}\) equals the original image \(f_{n,m}\).

**Statement 2:** *(Principle of superposition, or Paired Transform Slice Theorem I:)* The discrete image of size \(N \times N\), where \(N = 2^r\), \(r > 1\) can be decomposed by \((3N-2)\) direction images as

\[
f_{n,m} = \sum_{(p,s) \in J'_{N,N}} d_{n,m;p,s}
\]

\[
= \frac{1}{2N} \sum_{k=0}^{r-1} \frac{1}{2^k} \sum_{(p,s) \in 2^k J_{2^{r-k},2^{r-k}}} f'_{p,s,(np+ms) \mod N} + \frac{1}{N^2} f'_{0,0,0}.
\]
Since all direction images can be calculated directly from the projection data, we obtain the formula of reconstruction of the image from projections by its 2-D DPT, by using operations of addition/subtraction and division by powers of two. This theorem can be written in terms of the projection data in the following way.

**Corollary:** To reconstruct the discrete image

\[
  f_{n,m} = \sum_{(p,s) \in J_{N,N}'} d_{n,m,p,s}
\]

on the Cartesian grid \( N \times N \), where \( N = 2^r, r > 1 \), only \( 3N/2 \) projections are needed at the angles of the set

\[
  \Phi = \{ \varphi(p, s) = \arctg(s/p); \ (p, s) \in J_{N,N} \}.
\]

Figure 3 shows the tree image of size 256 \( \times \) 256 in part a, along the first 14 direction images, for the generators \((p, s) = (0, 1), (1, 1), (2, 1), (3, 1), \ldots, (11, 1), (254, 1), \) and \((255, 1)\) in parts b-o, respectively.

![Fig. 3. (a) Tree image and (b)–(o) 14 direction image components of the image. (All images have been scaled.)](image-url)
C. Principle of Superposition: General case

We now consider the image \( f_{n,m} \) of size \( N \times N \), when \( N = L^r \), \( r > 1 \), and \( L > 2 \) is a prime. In \( L \)-paired representation, the image is the set of \((L + 1)(L^r - 1) + 1\) splitting-signals,

\[
f_{T_p,s} = \{ f'_{p,s,0}; f'_{p,s,L^n}; f'_{p,s,2L^n}; \ldots; f'_{p,s,N/L-L^n} \}.
\]

which define the 2-D DFT at the corresponding sets

\[
T'_{p_1,p_2} = T'_{p_1,p_2,L} = \left\{ (mL + 1)p_1, (mL + 1)p_2; m = 0 : (N/L - 1) \right\},
\]

where \((mL + 1)p = (mL + 1)p \mod N\) for integers \( p \). The set of all generators \((p, s)\) of the paired transform components is defined as

\[
J'_{N,N} = \bigcup_{k=0}^{r-1} \bigcup_{j=1}^{L-1} \{ jL^k J_{L^r-k,L^r-k} \} \cup \{(0,0)\},
\]

and the subsets \( J_{L^r,L^k} \), for \( k = 1 : r \), are calculated by

\[
J_{L^r,L^k} = \bigcup_{p_2=0}^{L^k-1} (1, p_2) \bigcup_{p_1=0}^{L^{k-1}-1} (Lp_1, 1).
\]

The direction images are constructed similar to the \( L = 2 \) case. The incomplete 2-D DFT is composed only from the components of the 2-D DFT with frequency-points of the subset \( T'_{p,s} \),

\[
D_{p_1,s_1} = \begin{cases} 
F_{p_1,s_1}; & \text{if } (p_1, s_1) \in T'_{p,s} \\
0; & \text{otherwise.}
\end{cases}
\]

The image of this incomplete 2-D transform is composed only by values of the splitting-signal generated by \((p, s)\).

Accurate calculations of the inverse 2-D DFT of the incomplete transform, when defining the direction images, result in the following Principle of Superposition.

**Statement 3: Paired Transform Slice Theorem II:** The discrete image \( \{ f_{n,m} \} \) of size \( N \times N \), where \( N = L^r \), \( r > 1 \) can be decomposed by \((L + 1)(N - 1) + 1\) direction images as

\[
f_{n,m} = \sum_{(p,s) \in J'_{N,N}} d_{n,m;p,s}
\]

\[
= \sum_{k=0}^{r-1} \sum_{j=1}^{L-1} \sum_{(p,s) \in jL^k J_{N/L^k,N/L^k}} d_{n,m;p,s} + d_{n,m,0,0}
\]


where the direction images are calculated from the paired splitting-signals by

\[
d_{n,m;p,s} = \frac{1}{NL^{k+1}}f'_{p,s,L^k}(t \mod \frac{N}{L^{k+1}})W_{N/L^k}^{-t+t \mod \frac{N}{L^{k+1}}}^{N/L^{k+1}}
\]

and we denote \( t = (np/L^k + ms/L^k) \mod N/L^k \). \(^1\)

Each component of the paired transform can be calculated from the projection data. To reconstruct the discrete image on the Cartesian grid, only \((L+1)N/L\) projections are needed at the angles of the set

\[
\Phi = \{\varphi(p, s) = \arctg(s/p); (p, s) \in J_{N,N}\} = \{\arctg(1/p); p = 0 : (N - 1)\} \cup \{\arctg(Ls); s = 0 : (N/L - 1)\}.
\]

The codes for image decomposition by direction images, or image reconstruction from its projections by the tensor and paired transforms can be found in "Image Reconstruction" in our web page "http://www.fasttransforms.com"

**References**


\(^1\)The proof of this statement can be send by the request, A.M.G.