IV: Paired Representation and Resolution Map

This notes discuss the concept of the resolution map, as a result of uniting all direction images into $\log_2(N)$ series. Direction images are with respect to the paired representation. The paired representation leads to the image composition by a set of $3N-2$ direction images, which defines the directed multiresolution. In the resolution map, all different periodic components (or structures) of the image are packed into a matrix $N \times N$, which can be used for image processing in enhancement, filtration, and compression. The described direction image multiresolution is derived as a property of the two-dimensional discrete Fourier transform (2-D DFT), when it splits by 1-D DFTs. The image of size $N \times N$, where $N$ is a power of two.

We first describe the tensor and paired representation of the image, and, then, focus on the directional multiresolution, or resolution map.

I. The 2-D tensor transform

Let $F = \{ F_{p,s} \}$ be the two-dimensional $N \times N$-point DFT of the image $f_{n,m}$ which is defined on the Cartesian lattice $X_{N,N} = \{(n, m); n, m = 0 : (N-1)\}$. In definition of the 2-D DFT

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W^{np+ms}, \quad p, s = 0 : (N-1),$$

where $W = W_N = \exp(-i2\pi/N)$, we unite all components of the image at points $(n, m)$ at which the form $np + ms$ takes equal values in arithmetic modulo $N$. This simple form composed by frequency and spatial points is used to define the tensor representation of the image as a set of splitting-signals each of length $N$,

$$\chi_{N,N} : \{f_{n,m}\} \rightarrow \{\{f_{p,s,t}; t = 0 : (N-1)\}\}.$$  \hspace{1cm} (1)

The components of the splitting-signals are calculated by [Grigoryan, 1984]

$$f_{p,s,t} = \sum_{(n,m) \in X_{N,N}} \{f_{n,m}; np + ms = t \mod N\}.$$  \hspace{1cm} (2)

The splitting-signal carries the spectral information of the image in the group of frequency-points which are integer-multiple to $(p, s)$,

$$F_{kp \mod N, ks \mod N} = \sum_{t=0}^{N-1} f_{p,s,t} W^{kt}, \quad k = 0 : (N-1).$$  \hspace{1cm} (3)

Given integer $N > 1$, there are many ways to compose an irreducible covering of the whole set of frequency-points by such groups,

$$\bigcup_{(p,s) \in J_{N,N}} \{T_{p,s} = \{(kp \mod N, ks \mod N); k = 0 : (N-1)\}\} = X_{N,N},$$  \hspace{1cm} (4)

i.e. define different sets $J_{N,N}$ of frequency-generators $(p, s)$ for the tensor transformation in (1). The cardinality of such coverings, or a sets $J_{N,N}$ is one-valued. For instance, for $N = 8$, the cardinality of the set $J_{8,8}$ equals 12, and we can take the following set:

$$J_{8,8} = \{(1, 0), (1, 1), (1, 2), (1, 3), ..., (1, 7)\} \cup \{(0, 1), (2, 1), (4, 1), (6, 1)\}.$$
Frequency-generator (1, 0) can be changed by another generator, for example, (3, 0), in the set $J_{8,8}$. Such change of generators will not change the irreducible covering of the lattice $X_{8,8}$, because the cyclic groups $T_{1,0}$ and $T_{3,0}$ are equal. A permutation will occur in the tensor representation (1) of the image, namely a permutation of components in the splitting-signal \( \{ f_{1,0,t}; t = 0 : 7 \} \). When $N$ is a power of two $2^r$, $r > 1$, we consider the following set:
\[
J_{N,N} = \{(1,0), (1,1), (1,2), (1,3), \ldots , (1,N-1) \} \cup \{(0,1), (2,1), (4,1), (6,1), \ldots , (N-2,1) \}.
\] (5)

The set contains $3(2^{r-1})$ frequency-generators. This number shows the minimum number of splitting-signals, or 1-D DFTs required to calculate the 2-D DFT of the image by splitting-signals,
\[
f_{T_p,s} = \{ f_{p,s,0}, f_{p,s,1}, \ldots , f_{p,s,N-1} \}, \quad (p, s) \in J_{N,N}.
\]

Each splitting-signal is numbered by the corresponding cyclic group $T_{p,s}$, because the signal carries the spectral information of the image at frequency-points of this group.

II. The 2-D Paired Transform

When removing the redundancy in the tensor representation of the image of size $2^r \times 2^r$, the paired representation is derived [Grigoryan 1986], which allows for decomposing the image into different periodic structures in the spatial domain. Each of these periods is defined by a single frequency, and they also can be united by a specific subset of frequencies, which may include low and high frequencies as well. These subsets of frequencies are divided by different levels from low to high, as will be described in a moment. The paired representation provides direction multiresolution analysis of the image with localization both in point and set of frequencies. This view differs much from the known concept of the time-frequency localization in wavelets for multiresolution analysis of 1-D signals, which is also applied for 2-D images.

All components $f_{p,s,t}$ of the tensor transform are defined as linear integrals, or sums of the image, along the corresponding parallel directions. In other words, the tensor transform can be calculated from the projection data. Indeed, according to definition in (2), the components $f_{p,s,t}$ of the tensor transform are defined as linear integrals along the parallel lines
\[
xp + ys = t, \quad xp + ys = t + N, \quad \ldots, \quad xp + ys = t + (p + s - 1)N,
\]
that pass a set of points \((n, m)\) of the discrete lattice $X_{N,N}$ traced on the initial image. The characteristic function of this set of points
\[
V_{p,s,t} = \{(n, m); \; np + sm = t \; \text{mod} \; N \}
\]
is defined as
\[
\chi_{p,s,t}(n, m) = \begin{cases} 
1, & \text{if } np + ms = t \; \text{mod} \; N, \\
0, & \text{otherwise}.
\end{cases}
\] (6)

As an example, Figure 1 shows for the $N = 128$ case the gray scale images of four characteristic tensor functions (CTF) with triplet numbers \((1,3,4)\), \((1,3,22)\), and \((1,5,2)\), \((1,5,26)\) in parts a,b, and c,d, respectively. The function $\chi_{1,3,22}$ is a parallel shift of $\chi_{1,3,4}$, and the function $\chi_{1,5,26}$ is a parallel shift of $\chi_{1,5,2}$. In general, a series of $N$ periodic characteristics functions \( \{ \chi_{p,s,t}; t = 0 : (N - 1) \} \) are parallel shifted functions with respect to each other. One can say, that these functions are parallel shifted transformations of the first
function $\chi_{p,s,0}$. The functions of each series are defined by directions at the angle $\vartheta(p,s) = \arctg(p/s)$ to the horizontal $X$-axis (shown as a vertical on plots of Fig. 1).

Let $f = \{f_{n,m}\}$ be an image of size $N \times N$, where $N = 2^r$ and $r > 1$. The orthogonal 2-D discrete paired transformation represents uniquely the image as a family of $(3N - 2)$ splitting-signals [2]-[5]

$$\chi'_{N,N} : \{f_{n,m}\} \rightarrow \{f_{T'_{p,s}}, (p,s) \in J'_{N,N}\},$$

where each splitting-signal generated by a frequency $(p,s)$ is calculated by

$$f_{T'_{p,s}} = \{f'_{p,s,2^k t}; t = 0 : (N/2^{k+1} - 1), 2^k = g.c.d.(p,s)\}.$$  

The set of frequency-points, or generators $(p,s)$ is defined as

$$J'_{N,N} = \bigcup_{k=0}^{r-1} \{J_{N/2^k,N/2^k}\} \cup \{(0,0)\},$$

where we consider the following subsets:

$$J_{N/2^k,N/2^k} = \{2^k(1,s); s = 0 : (N/2^k - 1)\} \cup \{2^k(2p,1); p = 0 : (N/2^{k+1} - 1)\}.$$  

or

$$J_{N/2^k,N/2^k} = \{2^k(p,1); p = 0 : (N/2^k - 1)\} \cup \{2^k(1,2s); s = 0 : (N/2^{k+1} - 1)\}.$$  

The total number of the generators equals $(3N - 2)$.

Components of each such splitting-signal $f_{T'_{p,s}}$, which we call the paired splitting-signal, are calculated from the splitting-signal in the tensor representation by

$$f'_{p,s,2^k t} = f_{p,s,2^k t} - f_{p,s,2^k t + N/2}. \quad t = 0 : (N/2^{k+1} - 1).$$

These components are numbered by the triplets $(p,s,2^k t)$, where $(p,s)$ are the frequency-points that generate the splitting-signals. The time parameter $2^k t$ runs the interval $[0, N/2 - 1]$ with the step $2^k$ defined by the frequency $(p,s)$. The higher this step, the faster the time runs this interval, and therefore, the shorter the signal. The paired transformation is thus a transformation from the 2-D spatial domain into the (2-D frequency and 1-D time) domain. The structure of this 3-D domain is more complicated than in the tensor transformation case, when all splitting-signals are of the same length.
The paired signal carries the spectral information of the image at frequency-points of the following subset of the group $T_{p,s}$:

$$T'_{p,s} = \{(2m+1)p, (2m+1)s; \ m = 0 : (N/2^{k+1} - 1)\},$$

where we use the notation $l = l \mod N$, for integers $l$. Indeed, for a given frequency-point $(p, s)$, the following property holds for the $N \times N$-point DFT [Grigoryan, 1986]:

$$F_{(2m+1)p, (2m+1)s}^{N/2^{k+1} - 1} = \sum_{t=0}^{N/2^{k+1} - 1} (f_{p,s,2^kt} W_{N/2^k}^t) W_{N/2^k+1}^m, \ m = 0 : (N/2^{k+1} - 1).$$

(7)

The 2-D DFT at frequency-points of the subset $T'_{p,s}$ is defined by the $N/2^{k+1}$-point DFT of the splitting-signal $f_{T'_{p,s}}$ modified by the vector of twiddle coefficients $\{W_{N/2^k}^t; t = 0 : (N/2^{k+1} - 1)\}$.

The family of subsets $\{T'_{p,s}; (p, s) \in J'_{N,N}\}$ is a partition of the lattice $X_{N,N}$; these subsets are disjoint. Therefore, in paired representation, the 2-D DFT splits into a set of $(3N - 2)$ 1-D DFTs of different orders as follows:

$$N \times N$$-point 2-D DFT $\rightarrow \begin{cases} 3N/2 & \text{N/2-point DFTs}, \\ 3N/4 & \text{N/4-point DFTs}, \\ 3N/8 & \text{N/8-point DFTs}, \\ \ldots & \ldots \\ 6 & 2\text{-point DFTs}, \\ 3 & 1\text{-point DFTs}, \\ 1 & 1\text{-point DFT}. \end{cases}$

As an example, Figure 2 shows the bridge image $256 \times 256$ in part a, along with all splitting-signals of the image in b. Each series of splitting-signals of the same length is shown separately and in the order which is given in the definition of the set $J'_{256,256}$. The set of 1-D DFTs of the modified splitting-signals in b is shown in c.

.1 Complete system of 2-D paired functions

The 2-D discrete paired transform does not require multiplications. Its complete system of functions is defined as

$$\chi'_{p,s,2^kt}(n, m) = \begin{cases} 1, & \text{if } np + ms = 2^k t \mod N, \\ -1, & \text{if } np + ms = (2^k t + N/2) \mod N, \\ 0, & \text{otherwise}, \end{cases}$$

(8)

which is the difference of two characteristics functions $\chi_{p,s,2^kt}$ and $\chi_{p,s,2^kt+N/2}$. The paired function $\chi'_{p,s,2^kt}$ represents itself a 2-D plane wave, and the decomposition of the image by the paired functions is the decomposition of the image by plane waves. All basic paired functions are orthogonal. In the masks of paired functions, all “1”s and all “-1”s lie on the different but parallel lines, directions of which are defined by $(p, s)$. Inside each series of functions generated by frequency $(p, s)$,

$$\{\chi'_{p,s,2^kt}; t = 0, 1, 2, ..., N/2^{k+1} - 1, \ (2^k = \text{g.c.d.}(p, s))\},$$

the paired functions have the same frequency, direction, and they are parallel shifted functions with respect to each other.
A. Directed Multiresolution

The paired functions generated by the frequencies \((p, s)\) and \((2p, 2s)\), as well as \((4p, 4s), (8p, 8s), \ldots\), are defined by the same direction. Indeed the angles along which the coefficients “1” and “-1” are located in the masks equal the angle \(\vartheta = \arctg(p/s) = \arctg(2p/2s) = \arctg(4p/4s)\). The paired function \(\chi'_{2p,2s,2t}\) can be calculated by

\[
\chi'_{2p,2s,2t} = \chi_{2p,2s,2t} - \chi_{p,s,t} + \chi_{p,s,t+N/4} - \chi_{p,s,t+3N/4},
\]

and the components of the paired transform can be calculated through the tensor transform as follows:

\[
f'_{2p,2s,2t} = \frac{f'_{2p,2s,2t}}{f'_{2p,2s,2t}} = f_{p,s,t} - f_{p,s,t+N/4} + f_{p,s,t+N/2} - f_{p,s,t+3N/4}.
\]

Indeed, the following relation holds for tensor characteristic functions:

\[
\chi_{2p,2s,2t} = \chi_{2p,2s,2t} = \chi_{p,s,t} + \chi_{p,s,t+N/2}, \quad t = 0 : (N/2 - 1),
\]

where subscripts are considered modulo \(N\). Similarly, the components of the paired splitting-signals generated by frequencies \((4p, 4s), (8p, 8s), \ldots\) can be calculated from the same splitting-signal \(\{f_{p,s,t}; t = 0 : (N - 1)\}\). Although the complete family of paired functions are generated by \(3N - 2\) frequencies \((p, s)\), all these functions can be calculated from \(3N/2\) characteristic functions of the tensor transformation. In other words, all \(N^2\) paired functions are defined by directions at \(3N/2\) angles of the set

\[
\Psi_N = \{\arctg(1/s); s = 0 : (N - 1)\} \cup \{\arctg(2p); p = 0 : (N/2 - 1)\}.
\]
III. Paired transform direction images

According to Principle of superposition, the image \( \{ f_{n,m} \} \) can be composed by the direction images which are defined by splitting-signals in paired representation.

Let \((p, s)\) be the generator from the set \( J'_{N,N} \) and let \( 2^k = \text{g.c.d.}(p, s) \), where \( k \geq 0 \). We denote by \( D_{p_1,s_1} \) the incomplete 2-D DFT of the image, which is defined only at frequency-points of the subset \( T'_{p,s} \), i.e.

\[
D_{p_1,s_1} = D^{(p,s)}_{p_1,s_1} = \begin{cases} 
F_{p_1,s_1}; & \text{if } (p_1, s_1) \in T'_{p,s}, \\
0; & \text{otherwise}.
\end{cases}
\]  

For the first series of generators, when \( \text{g.c.d.}(p, s) = 1 \) (there are \( 3N/2 \) such generators that compose the subset \( J_{N,N} \)),

\[
d_{n,m} = d^{(p,s)}_{n,m} = \frac{1}{2N}f'_{p,s,(np+ms) \mod N}, \quad n, m = 0 : (N-1).
\]

The direction image \( N \times N \) is composed of \( N/2 \) values of the splitting-signal \( f'_{T'_{p,s}} \), which are placed on the image along the set of parallel lines \( np+ms = t \mod N, t = 0 : (N-1) \). As an example, Figure 3 shows for the bridge image \( 256 \times 256 \) the first nine direction images defined by frequencies \( (p, s) = (0, 1), (1, 1), \ldots, (8, 1) \) in parts a-i, respectively.

![Fig. 3. Nine first direction image components of the bridge image. (All images have been scaled.)](image)

According to the definition of the paired representation, the following property holds for its components: \( f'_{p,s,t+N/2} = -f'_{p,s,t} \) when \( t = 0 : (N/2 - 1) \). When \( (np+ms) \mod N \geq N/2 \), the values of \( f'_{p,s,t} \) are placed on the image with the minus sign. The composition of the direction image from the splitting-signal is simple. All values of this image can be calculated, for instance, by the following method of permutation of values of the splitting-signal.
For the \((k+1)\)th series of generators, i.e. such that \(\gcd(p, s) = 2^k\), when \(k \in \{1, 2, \ldots, r-1\}\), the direction image is defined as
\[
d_{n,m} = d_{n,m}^{(p,s)} = \frac{1}{2^{k+1}N} f'_{p,s,(np+ms) \mod N},
\]
where we denoted \(2^k t = (np + ms) \mod N\). The number of such generators in the set \(J'_{N,N}\) equals \(3N/2^{k+1}\).

As an example, Figure ?? shows the first nine direction images of the 2nd series, for the generators \((p, s) = (0, 2), (2, 2), \ldots, (14, 2), (16, 2)\) in parts a-i, respectively.

The last series of generators contains only the frequency \((0, 0)\) and the set \(T'_{0,0}\) equals \(\{(0, 0)\}\), and we have the following constant-image:
\[
d_{n,m} = d_{n,m}^{(0,0)} = \frac{1}{N^2} f_{0,0} = \frac{1}{N^2} \sum_{n_1=0}^{N-1} \sum_{m_1=0}^{N-1} f_{n_1,m_1} = \frac{1}{N^2} f_{0,0,0,0}.
\]

All \((3N-2)\) subsets \(T'_{p,s}\) with generators \((p, s) \in J'_{N,N}\), compose a partition of the lattice \(N \times N\). It means the sum of corresponding \((3N-2)\) incomplete 2-D DFTs equals the 2-D DFT of the image. It also means the sum of \((3N-2)\) direction images \(d_{n,m}^{(p,s)}\) equals the image \(f_{n,m}\).

**Superposition by paired direction images:** The discrete image can be composed from \((3N-2)\) direction images as
\[
f_{n,m} = \sum_{(p,s) \in J'_{N,N}} d_{n,m}^{(p,s)} = \frac{1}{2N} \sum_{k=0}^{r-1} \frac{1}{2^k} \sum_{(p,s) \in J_{2^r-k,2^r-k}} f'_{p,s,(np+ms) \mod N} + \frac{1}{N^2} f_{0,0,0,0}, \tag{10}
\]
\(n, m = 0 : (N - 1)\).

This is the formula of composition of the image by direction images and the formula of reconstruction of the image from its 2-D discrete paired transform.

**IV. Resolution Map of the Image**

From each image its periodic structures can be extracted and saved, and the image can be reconstructed from these structures. These structures are defined by the binary paired basis functions united by subsets and they can be packed in the unique matrix \(N \times N\), which is called the resolution map of the image.

To describe these periodic structures, we consider the concept of the \(k\)th series image, which is the sum of direction images corresponding to the subset of generators \(J_{N/2^k,N/2^k}\),
\[
S_{n,m}^{(k)} = \sum_{(p,s) \in J_{N/2^k,N/2^k}} d_{n,m}^{(p,s)}, k \in \{0, 1, \ldots, r-1\}.
\]

The \(r\)th series image is defined by the single image as \(S_{n,m}^{(r)} = d_{n,m}^{(0,0)} = E[f]\). Thus, the image can be calculated by
\[
f_{n,m} = \sum_{k=0}^{r} S_{n,m}^{(k)}, n, m = 0 : (N - 1).
\]

Figure 4 shows the first five series images for the bridge image 256 \(\times\) 256 in parts a through e.

One can recognize many details of the original image in the first series images, and some details on the second series. The series images in (b)-(e) have periodic structures with resolutions which increase
exponentially with the number of the series. We call the number $2^k$ the resolution of the $k$th series image. The first series image is the component of the image with the lowest resolution, and the $(r - 1)$th series image is the component of the image with the highest resolution. The constant image $S(r)$ has 0 resolution. The sum of only the first five series images with resolutions 1, 2, 4, 8, and 16 is shown in part f. To obtain the exact original bridge image, the remaining four series images with high resolutions 32, 64, 128, and resolution 0 should also be considered.

The detail analysis of the series images show that a unique set of periodic structures can be extracted from each series image in the following way. We first consider the first series image and its decomposition by periodic structures $N/2 \times N/2$. The set of generators $J_{N,N}$ is divided by three parts as

$$
J_{N,N}^{(1)} = \{(1, 2s); s = 0 : (N/2 - 1)\},
$$
$$
J_{N,N}^{(2)} = \{(2p, 1); p = 0 : (N/2 - 1)\},
$$
$$
J_{N,N}^{(3)} = \{(1, 2s + 1); s = 0 : (N/2 - 1)\}.
$$

The first two sets are symmetric to each other, and the third set of generators are unique. By using these three subsets of generators, the division of the first series image can be constructed, $S^{(0)} = P^{(0)} + N^{(0)} + U^{(0)}$, where

$$
P_{n,m}^{(0)} = \sum_{(p,s) \in J_{N,N}^{(1)}} d_{n,m}^{(p,s)}, \quad N_{n,m}^{(0)} = \sum_{(p,s) \in J_{N,N}^{(2)}} d_{n,m}^{(p,s)},
$$
$$
U_{n,m}^{(0)} = \sum_{(p,s) \in J_{N,N}^{(3)}} d_{n,m}^{(p,s)}.
$$

Figure 5 shows the image $P^{(0)}$ for the bridge image in part a, along with the images $N^{(0)}$ and $U^{(0)}$ in b and c, respectively. One can notice that each of these images is divided by four parts $N/2 \times N/2$ with similar structures, which can be used for composing the entire series image $S^{(0)}$. Indeed, the series image components $P^{(0)}, N^{(0)}$, and $U^{(0)}$ can be defined from their first quarters which are denoted by $P_1, N_1,$ and
$U_1$, respectively, as follows:

$P^{(0)} = \begin{bmatrix} P_1 & P_1 \\ -P_1 & -P_1 \end{bmatrix}$, $N^{(0)} = \begin{bmatrix} N_1 & -N_1 \\ N_1 & -N_1 \end{bmatrix}$,

$U^{(0)} = \begin{bmatrix} U_1 & -U_1 \\ -U_1 & U_1 \end{bmatrix}$. \hspace{1cm} (11)

Figure 6 shows the decomposition of the next series image $S^{(1)}$ for the bridge image. Each of these images is divided by four parts and each of these parts has one of the forms given in (11). Decomposition of the series image $S^{(2)}$ for the bridge image is shown in Figure 18. For this series, as well as the remaining series images $S^{(k)}$, $k = 3 : (r - 1)$, the decompositions similar to (11) are valid. Namely, the periods $N/2^{k+1} \times N/2^{k+1}$ of these images can be defined by the three quarters $P_{k+1}, N_{k+1},$ and $U_{k+1}$ in a way similar to the first series image.

As a result, the following resolution map (RM) associates with the image $f$:

\[
RM[f] = \begin{bmatrix}
P_1 & U_1 \\
N_1 & U_2 \\
N_2 & U_3 \\
N_3 & ... \\
\end{bmatrix}.
\]

(12)
This resolution map has the same size as the image and contains all periodic parts of the series images, i.e. all different periods by means of which the original image can be reconstructed. The RM represents itself the image packed by its periodic structures that correspond to a specific set of projections. Resolution map for the bridge image is shown in Figure 7.

Figure 7. (a) The bridge image and (b) its resolution map.

Figure 8 shows the boat image of size $512 \times 512$ in part a, along with the resolution map of this image in b. The resolution map can be used to change the resolution of the entire image, by processing direction images for desired directions. The reconstruction of the image from its resolution map is straightforward, because of the property of image component composition, which is given in (11).

It is clear, that each periodic structure in the resolution map can be also represented by its resolution map. In such a recursive way, the resolution map can be disassembled into small pieces, from which the whole image can be reconstructed. Some of these pieces can also be amplified to enhance the image, or can be removed for compression purpose. For instance, consider the following incomplete resolution maps of the
bridge image:

\[
\begin{array}{c|c|c}
P_1 & U_1 & N_1 \\
\hline
P_2 & U_2 & N_2 \\
\hline
P_3 & 0 & N_3 \\
\hline
\end{array}
\]

The removal of one of the parts of the series images with the lowest resolution, let say the part \( U_1 \), may damage much the image. However, this part can also be compressed after being removed from its resolution map, \( RM[U_1] \), a few series of images with high resolutions.

The resolution map packs all periodic structures of the image, and can be used for image processing, enhancement, filtration, and compression. The considered tensor and paired representations are 2-D frequency and 1-D time representations of the image; they are unique and have been derived from the core of the 2-D DFT. They provide the multiresolution of the image and this multiresolution is directed by a certain set of angles.

**References**


